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LETTER TO THE EDITOR

Critical points of two-dimensional Ising models

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Abstract. A general method for computing critical points of planar two-dimensional Ising models is outlined. As examples we compute the critical points of a pair of archimedean lattices.

This letter was prompted by a recent article (Sykes *et al* 1974) concerning critical percolation probabilities (p_c) of closely similar lattices. It was stated in particular that small differences in computed series expansions for percolation probabilities cannot positively rule out the conjecture (Neal 1972) that the pair of archimedean lattices shown in figure 1 have identical p_c.

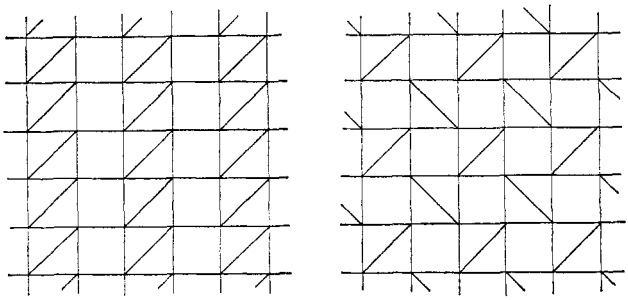

Figure 1

To shed some light on this problem we have computed the exact critical temperatures for the Ising problems on lattices 1(*a*) and 1(*b*). In terms of the variable $z = \exp(2K)$, $K = J/kT$, J being the nearest neighbour coupling constant we find critical values

$$z_c = \begin{cases} 2 & \text{for } 1(a) \\ 1{\cdot}9807308\ldots & \text{for } 1(b) \end{cases} \tag{1}$$

suggesting that the corresponding p_c should be close but probably not identical.

Our method, outlined below, used to obtain (1) is based on the combinatorial approach and works for any regular two-dimensional planar lattice. It also reveals an interesting connection between lattice structure and critical points.

After this work was completed an article (Syozi 1955) was brought to our attention which contained a method, based on the algebraic approach, for computing Ising model critical points. In particular Syozi computed the critical points of lattices 1(*a*)

and 1(*b*) finding $z_c = 2$ and 2·36 . . . respectively. Syozi's critical equation for 1(*b*) is in fact incorrect and should read

$$(2\cosh 2K_c - 1)(\cosh 2K_c - 1)^{-1} = 2\cosh^2 2K_c \exp(2K_c). \tag{2}$$

The solution of (2) agrees precisely with the value given in (1).

Our method for computing Ising model critical points is based on Vdovichenko's version of the combinatorial approach (Vdovichenko 1965) which expresses the partition function Z_N for a planar lattice of N points and coordination number q as

$$Z_N = 2^N(\cosh K)^{Nq/2} \exp\left(-\frac{1}{2}\sum_{r=0}^{\infty} b_r v^r/r\right) \tag{3}$$

where

$$v = \tanh K \tag{4}$$

and b_r is the sum over all single, directed loops with r bonds, each bond being weighted by a factor $\exp(i\phi/2)$ where ϕ is the change of direction in going over to the next bond.

Further reduction (for details see Vdovichenko 1965 or Stanley 1971) yields

$$Z_N \sim 2^N(\cosh K)^{Nq/2} \exp\left(\frac{1}{2}\sum_i \ln(1 - v\lambda_i)\right), \qquad \text{as } N \to \infty \tag{5}$$

where λ_i are the eigenvalues of a transition matrix A, each of its elements being the 'probability matrix' W_1 (the construction of which is described below) of a transition between adjacent lattice points.

For our purposes we require only the maximum eigenvalue λ_{max} (in the thermodynamic limit) since from (5) it is clear that the critical point is given by

$$v_c = \tanh K_c = (\lambda_{max})^{-1}. \tag{6}$$

Inspection of the diagonalizing procedures for the transition matrix A reveals that the required λ_{max} is simply the maximum eigenvalue of the probability matrix W_1. This result is valid for any planar two-dimensional lattice, the only problem now being the construction of W_1.

By way of illustration, consider a unit cell of the square lattice surrounding the point P, with nearest-neighbour lattice points P_1–P_4 and bonds oriented in the directions 1–4 as shown in figure 2. To leave P in the direction 1 it is clear that we must either come from P_2 in the direction 4 turning through $\pi/2$ at P, from P_3 in the

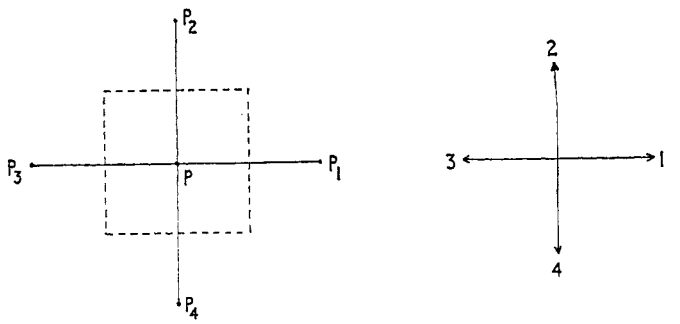

Figure 2

direction 1 with no turn, or from P_4 in the direction 2 turning through $-\pi/2$ at P. Recalling that a turn through an angle ϕ is weighted by a factor $\exp(i\phi/2)$ the above can be expressed probabilistically, or symbolically as

$$(1)' = (1)+\alpha^{-1}(2)+0(3)+\alpha(4)$$

where $\alpha = \exp(i\pi/4)$.

Similarly by considering departures from P in the remaining three directions we obtain

$$(2)' = \alpha(1)+(2)+\alpha^{-1}(3)+0(4)$$
$$(3)' = 0(1)+\alpha(2)+(3)+\alpha^{-1}(4)$$
$$(4)' = \alpha^{-1}(1)+0(2)+\alpha(3)+(4)$$

yielding the matrix

$$W_1 = \begin{pmatrix} 1 & \alpha^{-1} & 0 & \alpha \\ \alpha & 1 & \alpha^{-1} & 0 \\ 0 & \alpha & 1 & \alpha^{-1} \\ \alpha^{-1} & 0 & \alpha & 1 \end{pmatrix}. \tag{7}$$

The matrix (7) is doubly stochastic with $1+\alpha^{-1}+\alpha = 1+\sqrt{2}$ as the maximum eigenvalue. Hence from (6) we obtain the well known result: $v_c = (1+\sqrt{2})^{-1}$ for the square lattice.

As a further illustration, consider the honeycomb lattice as shown in figure 3.

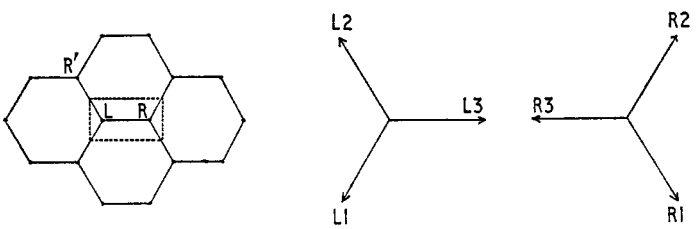

Figure 3

Arguing as before we see that to leave L in the direction L1 say, we must have either come from R in the direction R3 or from R′ in the unit cell above in the direction R1, turning respectively through angles $+\pi/3$ and $-\pi/3$. That is

$$(L1)' = \alpha(R3)+\alpha^{-1}(R1), \qquad \alpha = \exp(i\pi/6).$$

Five similar equations can be written down by inspection giving (with rows and columns in order L1 L2 L3 R1 R2 R3)

$$W_1 = \begin{pmatrix} 0 & 0 & 0 & \alpha^{-1} & 0 & \alpha \\ 0 & 0 & 0 & 0 & \alpha & \alpha^{-1} \\ 0 & 0 & 0 & \alpha & \alpha^{-1} & 0 \\ \alpha & 0 & \alpha^{-1} & 0 & 0 & 0 \\ 0 & \alpha^{-1} & \alpha & 0 & 0 & 0 \\ \alpha^{-1} & \alpha & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

This matrix is also doubly stochastic with maximum eigenvalue $\alpha+\alpha^{-1} = \sqrt{3}$ giving $v_c = 1/\sqrt{3}$.

At this stage the two archimedean lattices 1(*a*) and 1(*b*) and more complicated structures should present no difficulty. Lattice 1(*a*) for example with unit cell and directions indicated in figure 4 yields $(L1)' = \alpha^3(L4)+\alpha(R3)+\alpha^{-1}(L2)+\alpha^{-3}(R5)$ and

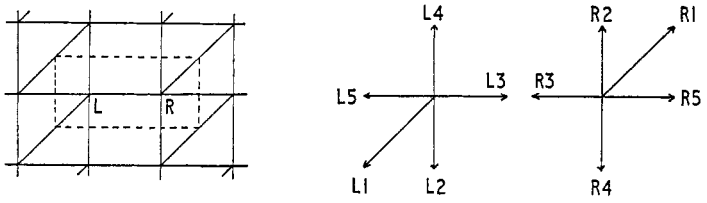

Figure 4

nine similar equations by inspection with $\alpha = \exp(i\pi/8)$, giving

$$W_1 = \begin{pmatrix}
0 & \alpha^{-1} & 0 & \alpha^3 & 0 & 0 & 0 & \alpha & 0 & \alpha^{-3} \\
0 & 1 & 0 & 0 & 0 & \alpha^{-3} & 0 & \alpha^2 & 0 & \alpha^{-2} \\
0 & \alpha^2 & 0 & \alpha^{-2} & 0 & \alpha^{-1} & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & \alpha & 0 & \alpha^{-2} & 0 & \alpha^2 \\
0 & \alpha^{-2} & 0 & \alpha^2 & 0 & \alpha^3 & 0 & 1 & 0 & 0 \\
0 & 0 & \alpha & 0 & \alpha^{-3} & 0 & \alpha^{-1} & 0 & \alpha^3 & 0 \\
\alpha^{-3} & 0 & \alpha^2 & 0 & \alpha^{-2} & 0 & 1 & 0 & 0 & 0 \\
\alpha^{-1} & 0 & 0 & 0 & 1 & 0 & \alpha^2 & 0 & \alpha^{-2} & 0 \\
\alpha & 0 & \alpha^{-2} & 0 & \alpha^2 & 0 & 0 & 0 & 1 & 0 \\
\alpha^3 & 0 & 1 & 0 & 0 & 0 & \alpha^{-2} & 0 & \alpha^2 & 0
\end{pmatrix} \tag{9}$$

(with rows and columns in order L1–L5, R1–R5). The maximum eigenvalue of (9) was found to be 3 giving $v_c = 1/3$ or $z_c = 2$ (cf equation (1)).

In general it is clear that the dimension of the matrix W_1 will be the number of lattice points in a unit cell times the coordination number of the lattice. For lattice 1(*b*) there are four sites per unit cell giving a 20×20 matrix W_1. The maximum eigenvalue of the matrix was found on a computer to be 3·0392955 giving $z_c = 1{\cdot}9807308$.

We remark in conclusion that the above method can be easily generalized to treat anisotropic Ising lattices by including weights $\tanh(J_i/kT)$ for bonds connecting spins interacting with strength J_i. The critical point is then found by equating the maximum eigenvalue of the resulting matrix to unity.

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